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Some existence and stability results for integro-differential equations by ψ-Hilfer fractional derivative

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Abstract. In this paper, we discuss the existence, uniqueness and stability differential equation with ψ-Hilfer fractional derivative. The arguments are based upon Schauder fixed point theorem, Banach contraction principle and ulam type stability.

Keywords: ψ-Hilfer fractional derivative, Existence, Stability.

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1 Introduction

Fractional calculus is a generalization of regular differentiation and integration to arbitrary order (non-integer). In latest years, fractional differential equations(FDEs) rise up certainly in various fields which include rheology, fractals, chaotic dynamics, control theory, signal processing, bioengineering and biomedical applications, and many others. Theory of FDEs has been extensively studied by many authors [5, 3, 7, 6, 8, 12]. Recentely, much attention has been paid to existence results for the integro-differential equation see [1, 2, 4]. Rassias established the Hyers-Ulam stability of linear and nonlinear mapping. This outcome of Rassias attracted many investigators worldwide who began to be stimulated to investigate the stability problems of differential equations [9, 10, 18, 19]. The fractional Ulam stability introduced by Wang [18, 19] and Ibrahim [13]-[16]. In this work, we investigate the existence, uniqueness and stability of fractional differential equations involving ψ-Hilfer fractional derivative which initiated by J. Vanterler da C. Sousa and E. Capelas de Oliveira in [17]. ψ-Hilfer fractional derivative unifies many fractional derivative and a note on the transformation can be found in [17].

Consider the integro-differential equation involving ψ-Hilfer fractional derivative of the form

$$\begin{cases} D_{a^+}^{\alpha,\beta;\psi}x(t) = f(t,x(t),\int_a^t h(t,s,x(s))ds), \quad t \in J := (a,b], \\ I_{a^+}^{1-\gamma;\psi}x(a) = x_a, \quad \gamma = \alpha+\beta-\alpha\beta, \end{cases} \tag{1.1}$$

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where ${}^{\rho}D_{a+}^{\alpha,\beta;\psi}$ is ψ-Hilfer fractional derivative of order α and type β and $I_{a+}^{1-\gamma;\psi}$ is ψ-fractional integral of order $1-\gamma$, where $f : J \times R \times R \to R$ $h : \Delta \times R \to R$ are continuous. Here, $\Delta = \{(t,s) : a \le s \le t \le b\}$. For brevity let us take

$$Hx(t) = \int_a^t h(t,s,x(s))ds.$$

The paper is organized as follows. In section 2, we present notations and definition used throughout the paper. In Section 3, we discuss the existence and uniqueness results for integro-differential equation Schauder fixed-point theorem and contraction principle. In Section 4, four types of Ulam stability, namely Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias and generalized Ulam-Hyers-Rassias stability is discussed.

2 Preliminary

In this section, we recall some definitions and results from fractional calculus. The following observations are taken from [6, 11]. Throughout this paper, let $C[a,b]$ a space of continuous functions from J into R with the norm

$$\|x\| = \sup\{|x(t)| : t \in J\}.$$

The weighted space $C_{\gamma,\psi}[a,b]$ of functions f on $(a,b]$ is defined by

$$C_{\gamma,\psi}[a,b] = \{f : (a,b] \to R : (\psi(t)-\psi(a))^{\gamma} f(t) \in C[a,b]\}, 0 \le \gamma < 1,$$

with the norm

$$\|f\|_{C_{\gamma,\psi}} = \left\|(\psi(t)-\psi(a))^{\gamma} f(t)\right\|_{C[a,b]} = \max_{t\in J}\left|(\psi(t)-\psi(a))^{\gamma} f(t)\right|.$$

The weighted space $C^n_{\gamma,\psi}[a,b]$ of functions f on $(a,b]$ is defined by

$$C^n_{\gamma,\psi}[a,b] = \left\{f : J \to R : f(t) \in C^{n-1}[a,b]; f(t) \in C_{\gamma,\psi}[a,b]\right\}, 0 \le \gamma < 1, 0 \le \gamma < 1$$

with the norm

$$\|f\|_{C^n_{\gamma,\psi}[a,b]} = \sum_{k=0}^{n-1}\left\|f^k\right\|_{C[a,b]} + \|f^n\|_{C_{\gamma,\psi}[a,b]}.$$

For $n = 0$, we have, $C^0_{\gamma}[a,b] = C_{\gamma}[a,b]$.

Definition 2.1. The left-sided fractional integral of a function f with respect to another function ψ on [a, b] is defined by

$$\left(I_{a^+}^{\alpha;\psi}\right) f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1} f(s)ds,\ t > a. \tag{2.1}$$

Definition 2.2. Let $\psi'(x) \ne 0$ $(-\infty \le < t < b < \infty)$ and $\alpha > 0$, $n \in N$. The Riemann-Liouville fractional derivative of a function f with respect to ψ of order α correspondent to the Riemann-Liouville, is defined by

$$\left(D_{a^+}^{\alpha;\psi} f\right)(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \int_a^t \psi'(s)\left(\psi(t)-\psi(s)\right)^{n-\alpha-1} f(s)ds, \tag{2.2}$$

where $n = [\alpha] + 1$.

Definition 2.3. Let $\alpha > 0$, $n \in N$, $I = [a,b]$ is the interval $(-\infty \leq < t < b < \infty)$, $f, \psi \in C^n([a,b], R)$ two functions such that ψ is increasing and $\psi'(x) \neq 0$, for all $x \in I$. The left ψ-Caputo derivative of f of order α is given by

$$\left(D_{a^+}^{\alpha;\psi} f\right)(t) = I_{a^+}^{n-\alpha;\psi} \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^n f(t) \tag{2.3}$$

where $n = [\alpha] + 1$ for $\alpha \notin N$ and $\alpha = n$ for $\alpha \in N$.

Definition 2.4. The ψ-Hilfer fractional derivative of function f oforder α is given by,

$$D_{a^+}^{\alpha,\beta;\psi} f(t) = I_{a^+}^{\beta(1-\alpha);\psi} \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right) I_{a^+}^{(1-\beta)(1-\alpha);\psi} f(t). \tag{2.4}$$

The ψ-Hilfer fractional derivative as above defined, can be written in the following

$$D_{a^+}^{\alpha,\beta;\psi} f(t) = I_{a^+}^{\gamma-\alpha;\psi} D_{a^+}^{\gamma;\psi} f(t).$$

Lemma 2.5. *Let $\alpha, \beta > 0$, Then we have the following semigroup property*

$$(I_{a^+}^{\alpha;\psi} I_{a^+}^{\beta;\psi} f)(t) = (I_{a^+}^{\alpha+\beta;\psi})(t),$$

and

$$(D_{a^+}^{\alpha;\psi} I_{a^+}^{\alpha;\psi} f)(t) = f(t).$$

Lemma 2.6. *Let $\alpha, \beta > 0$, and*

1. *If $f(x) = (\psi(t)\psi(a))^{\beta-1}$, then*

$$I_{a^+}^{\alpha;\psi} (\psi(t) - \psi(a))^{\beta-1} (t) = \frac{\Gamma(\beta)}{(\alpha+\beta)} (\psi(t) - \psi(a))^{\alpha+\beta-1}.$$

2. *If $g(x) = (\psi(t)\psi(a))^{\alpha-1}$, then*

$$D_{a^+}^{\alpha;\psi} (\psi(t) - \psi(a))^{\alpha-1} (t) = 0.$$

Lemma 2.7. *Let $0 < \alpha < 1$. If $f \in Cn[a,b]$, then*

$$\left(I_{a^+}^{\alpha;\psi} D_{a^+}^{\alpha;\psi}\right)(t) = f(t) - \frac{\left(I_{a+}^{1-\alpha;\psi} f\right)(a)}{\Gamma(\alpha)} (\psi(t) - \psi(a))^{\alpha-1},$$

for all $x \in (a,b]$.

Lemma 2.8. *Let $n - 1 \leq \gamma < n$ and $f \in C_\gamma[a,b]$. Then*

$$\left(I_{a^+}^{\alpha;\psi} f\right)(a) = \lim_{t \to a^+} \left(I_{a^+}^{\alpha;\psi}\right) f(t) = 0.$$

Here we present the following weighted space as follows

$$C_{1-\gamma;\psi}^{\alpha,\beta}[a,b] = \left\{f \in C_{1-\gamma;\psi}[a,b], D_{a^+}^{\alpha,\beta;\psi} f \in C_{\gamma;\psi}[a,b]\right\}$$

and

$$C_{1-\gamma;\psi}^{\gamma}[a,b] = \left\{f \in C_{1-\gamma;\psi}[a,b], D_{a^+}^{\gamma;\psi} f \in C_{1-\gamma;\psi}[a,b]\right\}.$$

It is obvious that

$$C_{1-\gamma;\psi}^{\gamma}[a,b] \subset C_{1-\gamma;\psi}^{\alpha,\beta}[a,b].$$

Lemma 2.9. *Let $0 < \alpha < 1$, $0 \leq \beta \leq 1$ and $\gamma = \alpha + \beta - \alpha\beta$. If $C^{\gamma}_{1-\gamma,\psi}[a,b]$, then*

$$I^{\gamma;\psi}_{a^+} D^{\gamma;\psi}_{a^+} f = I^{\alpha;\psi}_{a^+} D^{\alpha,\beta;\psi}_{a^+} f \tag{2.5}$$

and

$$D^{\gamma;\psi}_{a^+} I^{\alpha;\psi}_{a^+} f = D^{\beta(1-\alpha);\psi}_{a^+} f. \tag{2.6}$$

Lemma 2.10. *Let $f \in L^1(a,b)$. If $D^{\beta(1-\alpha);\psi}_{a^+} f$ exists on $L^1(a,b)$, then*

$$D^{\alpha,\beta;\psi}_{a^+} I^{\alpha;\psi}_{a^+} f = I^{\beta(1-\alpha);\psi}_{a^+} D^{\beta(1-\alpha);\psi}_{a^+} f.$$

Lemma 2.11. *Let $f \in C^1[a,b]$, $\alpha > 0$ and $0 \leq \beta \leq 1$, we have*

$$D^{\alpha,\beta;\psi}_{a^+} I^{\alpha;\psi}_{a^+} f = f.$$

Lemma 2.12. *Suppose $\alpha > 0$, $a(t)$ is a nonnegative function locally integrable on $a \leq t < b$ (some $b \leq \infty$), and let $g(t)$ be a nonnegative, nondecreasing continuous function defined on $a \leq t < b$, such that $g(t) \leq K$ for some constant K. Further let $x(t)$ be a nonnegative locally integrable on $a \leq t < b$ function with*

$$|x(t)| \leq a(t) + g(t) \int_a^t \psi'(s) \left(\psi(t) - \psi(s)\right)^{\alpha-1} x(s) ds, \quad t \in J$$

with some $\alpha > 0$. Then

$$|x(t)| \leq a(t) + \int_a^t \left[\sum_{n=1}^{\infty} \frac{(g(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)} \psi'(s) \left(\psi(t) - \psi(s)\right)^{n\alpha-1} \right] x(s) ds, \quad a \leq t < b.$$

Proof. The proof is similar to Theorem 1 in [20]. □

Lemma 2.13. *Let $\gamma = \alpha + \beta - \alpha\beta$, where $0 < \alpha < 1$ and $0 \leq \beta \leq 1$. If $f : J \times R \times R \to R$ is a function such that $f(\cdot, x(\cdot)) \in C_{1-\gamma}[a,b]$ for all $x \in C_{1-\gamma}[a,b]$. A function $x \in C^{\gamma}_{1-\gamma}[a,b]$ is the solution of fractional initial value problem*

$$\begin{cases} D^{\alpha,\beta;\psi}_{a_+} x(t) = f(t, x(t), Hx(t)), \ 0 < \alpha < 1, \ 0 \leq \beta \leq 1, \\ I^{1-\gamma;\psi}_{a_+} x(a) = x_a, \end{cases}$$

if and only if x satisfies the following Volterra integral equation

$$x(t) = \frac{x_a}{\Gamma(\gamma)} \left(\psi(t) - \psi(a)\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\psi(t) - \psi(s)\right)^{\alpha-1} f(s, x(s), Hx(s)) ds. \tag{2.7}$$

3 Existence and uniqueness

We make the following hypotheses to prove our main results.

(H1) Let $f : J \times R \to R$ be a function such that $f(\cdot, x(\cdot)) \in C_{1-\gamma,\psi}[J,R]$ for any $x \in C_{1-\gamma}[J,R]$. For all $x, y \in R$, there exists a positive constant $L > 0$ such that

$$|f(t,x) - f(t,y)| \leq L\,|x - y|\,.$$

(H2) : Let $h : \Delta \times R \to R$ be continuous and there exists a constant $H > 0$, such that

$$\int_0^t |h(t,s,x) - h(t,s,y)| \leq H |x - y| .$$

(H3) Let $f : J \times R \to R$ a function and there exists a constant M, N such that

$$|f(t,x)| \leq M |x| + N, \ \forall \ t \in J, \ x \in R.$$

Theorem 3.1. *Assume that [H1] and [H2] are satisfied. Then,* (1.1) *has at least one solution.*

Proof. Consider the operator $N : C_{1-\gamma,\psi}[a,b] \to C_{1-\gamma,\psi}[a,b]$. The equivalent integral equation (2.7) which can be written in the operator form

$$x(t) = Nx(t)$$

where

$$(Nx)(t) = \frac{x_a}{\Gamma(\gamma)} (\psi(t) - \psi(a))^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} f(s, x(s), Hx(s))ds. \quad (3.1)$$

Consider the ball

$$B_r = \left\{ x \in C_{1-\gamma,\psi}[a,b] : \|x\| \leq r \right\}$$

It is obvious that the operator N is well defined. Clearly, the fixed points of the operator N are solutions of the problem. For any $x \in C_{1-\gamma,\psi}[a,b]$ and each $t \in J$ we have,

$$\begin{aligned}
&\left| (Nx)(t) (\psi(t) - \psi(a))^{1-\gamma} \right| \\
&= \left| \frac{x_a}{\Gamma(\gamma)} + \frac{1}{\Gamma(\alpha)} (\psi(t) - \psi(a))^{1-\gamma} \int_a^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} f(s, x(s), Hx(s))ds \right| \\
&\leq \frac{x_a}{\Gamma(\gamma)} + \frac{1}{\Gamma(\alpha)} (\psi(t) - \psi(a))^{1-\gamma} \int_a^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} |f(s, x(s))| \, ds \\
&\leq \frac{x_a}{\Gamma(\gamma)} + \frac{1}{\Gamma(\alpha)} (\psi(t) - \psi(a))^{1-\gamma} \int_a^t \psi'(s) (\psi(t) - \psi(a))^{\alpha-1} (M |x(s)| + N) \, ds \\
&\leq \frac{x_a}{\Gamma(\gamma)} + \frac{M}{\Gamma(\alpha)} (\psi(t) - \psi(a))^{1-\gamma} (\psi(t) - \psi(a))^{\alpha+\gamma-1} B(\gamma, \alpha) \|x\|_{C_{1-\gamma,\psi}} \\
&+ \frac{N}{\Gamma(\alpha+1)} (\psi(t) - \psi(a))^{1-\gamma} (\psi(t) - \psi(a))^{\alpha} \\
&\leq \frac{x_a}{\Gamma(\gamma)} + \frac{1}{\Gamma(\alpha)} (\psi(b) - \psi(a))^{\alpha} B(\gamma, \alpha) \|x\|_{C_{1-\gamma,\psi}} + \frac{N}{\Gamma(\alpha+1)} (\psi(b) - \psi(a))^{\alpha+1-\gamma} .
\end{aligned}$$

This proves that N transforms the ball $B_r = \left\{ x \in C_{1-\gamma,\psi}[a,b] : \|x\|_{C_{1-\gamma,\psi}} \leq r \right\}$ into itself. The proof is divided into several steps.

Step 1: The operator N is continuous.

Let x_n be a sequence such that $x_n \to x$ in $C_{1-\gamma,\psi}[a,b]$. Then for each $t \in J$,

$$\begin{aligned}
&\left|((Nx_n)(t) - (Nx)(t))\,(\psi(t)-\psi(a))^{1-\gamma}\right| \\
\leq &\left|\frac{(\psi(t)-\psi(a))^{1-\gamma}}{\Gamma(\alpha)}\int_a^t \psi^{'}(s)\,(\psi(t)-\psi(s))^{\alpha-1} f(s,x_n(s),Hx_n(s))ds\right. \\
&\left.-\frac{(\psi(t)-\psi(a))}{\Gamma(\alpha)}\int_a^t \psi^{'}(s)\,(\psi(t)-\psi(s))^{\alpha-1} f(s,x(s),Hx(s))ds\right| \\
\leq &\frac{(\psi(t)-\psi(a))^{1-\gamma}}{\Gamma(\alpha)}\int_a^t \psi^{'}(s)\,(\psi(t)-\psi(s))^{\alpha-1}\left|f(s,x_n(s),Hx_n(s)) - f(s,x(s),Hx(s))\right| ds \\
\leq &\frac{(\psi(t)-\psi(a))^{1-\gamma}}{\Gamma(\alpha)}\,(\psi(t)-\psi(a))^{\alpha+\gamma-1} B(\gamma,\alpha)\,\|f(\cdot,x_n(\cdot),Hx_n(\cdot)) - f(\cdot,x(\cdot),Hx(\cdot))\|_{C_{1-\gamma,\psi}},
\end{aligned}$$

which implies

$$\|Nx_n - Nx\|_{C_{1-\gamma,\psi}} \leq B(\gamma,\alpha)\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha)}\,\|f(\cdot,x_n(\cdot),Hx_n(\cdot)) - f(\cdot,x(\cdot),Hx(\cdot))\|_{C_{1-\gamma,\psi}}.$$

It implies that N is continuous.
Step 2: $N(B_r)$ is uniformly bounded.
It is clear that $N(B_r) \subset B_r$ is bounded.
Step 3: $N(B_r)$ is relatively compact.
It follows from $N(B_r) \subset B_r$ that $N(B_r)$ is uniformly boundeed. Moreover, to show that N is an equicontinuous operator. Let $t_1, t_2 \in J, t_1 < t_2, B_r$ be a bounded set of $C_{1-\gamma,\rho}[a,b]$. Then,

$$\begin{aligned}
&|((Nx)(t_1) - (Nx)(t_2))| \\
&\leq \frac{x_a}{\Gamma(\gamma)}\left|(\psi(t_1)-\psi(a))^{\gamma-1} - (\psi(t_2)-\psi(a))^{\gamma-1}\right| \\
&+ \frac{B(\gamma,\alpha)}{\Gamma(\alpha)}\left((\psi(t_1)-\psi(a))^{\alpha+\gamma-1} - (\psi(t_2)-\psi(a))^{\alpha+\gamma-1}\right)\|f\|_{C_{1-\gamma,\psi}}
\end{aligned}$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero. As a consequence of claim 1 to 3, together with Arzela-Ascoli theorem, we can conclude that $N : C_{1-\gamma,\psi}[a,b] \to C_{1-\gamma,\psi}[a,b]$ is continuous and completely continuous.

□

Theorem 3.2. *Assume that hypothesis (H1) is fulfilled. If*

$$\frac{L(1+H)}{\Gamma(\alpha)}B(\gamma.\alpha)\,(\psi(b)-\psi(a))^{\alpha} < 1$$

then, Eq. (1.1) *has unique solution.*

4 Stability Analysis

Next, we shall give the definitions and the criteria of Ulam-Hyers stability and Ulam-Hyers-Rassias stability for impulsive fractional differential Eq.(1.1). Let $\epsilon > 0$ be a positive real number and $\varphi : J \to R^+$ be a continuous function. We consider the following inequalities

$$\left|D_{a_+}^{\alpha,\beta;\psi} z(t) - f(t, z(t), Hz(t))\right| \le \epsilon, \quad t \in J, \tag{4.1}$$

$$\left|D_{a_+}^{\alpha,\beta;\psi} z(t) - f(t, z(t), Hz(t))\right| \le \epsilon\varphi(t), \quad t \in J, \tag{4.2}$$

$$\left|D_{a_+}^{\alpha,\beta;\psi} z(t) - f(t, z(t), Hz(t))\right| \le \varphi(t), \quad t \in J. \tag{4.3}$$

Definition 4.1. The Eq. (1.1) is Ulam-Hyers stable if there exists a real number $C_f > 0$ such that for each $\epsilon > 0$ and for each solution $z \in C_{1-\gamma,\psi}[a,b]$ of the inequality (4.1) there exists a solution $x \in C_{1-\gamma,\psi}[a,b]$ of Eq. (1.1) with

$$|z(t) - x(t)| \le C_f\, \epsilon, \quad t \in J.$$

Definition 4.2. The Eq. (1.1) is generalized Ulam-Hyers stable if there exist $\varphi \in C_{1-\gamma,\psi}[a,b]$, $\varphi_f(0) = 0$ such that for each solution $z \in C_{1-\gamma,\psi}[a,b]$ of the inequality (4.1) there exists a solution $x \in C_{1-\gamma,\psi}[a,b]$ of Eq. (1.1) with

$$|z(t) - x(t)| \le \varphi_f\, \epsilon, \quad t \in J.$$

Definition 4.3. The Eq. (1.1) is Ulam-Hyers-Rassias stable with respect to $\varphi \in C_{1-\gamma,\psi}[a,b]$ if there exists a real number $C_{f,\varphi} > 0$ such that for each $\epsilon > 0$ and for each solution $z \in C_{1-\gamma,\psi}[a,b]$ of the inequality (4.2) there exists a solution $x \in C_{1-\gamma,\psi}[a,b]$ of Eq. (1.1) with

$$|z(t) - x(t)| \le C_{f,\varphi}\, \epsilon\varphi(t), \quad t \in J.$$

Definition 4.4. The Eq. (1.1) is generalized Ulam-Hyers-Rassias stable with respect to $\varphi \in C_{1-\gamma,\psi}[a,b]$ if there exists a real number $C_{f,\varphi} > 0$ such that for each solution $z \in C_{1-\gamma,\psi}[a,b]$ of the inequality (4.3) there exists a solution $x \in C_{1-\gamma,\psi}[a,b]$ of Eq. (1.1) with

$$|z(t) - x(t)| \le C_{f,\varphi}\varphi(t), \quad t \in J.$$

Remark 4.5. Clearly,
1. Definition 4.1 $\Rightarrow$ Definition 4.2.
2. Definition 4.3 $\Rightarrow$ Definition 4.4.
3. Definition 4.3 for $\varphi(t) = 1 \Rightarrow$ Definition 4.1

Remark 4.6. A function $z \in C_{1-\gamma,\psi}[a,b]$ is a solution of the inequality (4.1) if and only if there exists a function $g \in C_{1-\gamma,\psi}[a,b]$ such that

$$\left|D_{a_+}^{\alpha,\beta;\psi} z(t) - f(t, z(t), Hz(t))\right| \le \epsilon, \quad t \in J,$$

if and only if there exist a function $g \in C_{1-\gamma,\psi}[a,b]$ such that

(i) $|g(t)| \le \epsilon, t \in J$.

(ii) $D_{a_+}^{\alpha,\beta;\psi} z(t) = f(t, z(t), Hz(t)) + g(t),\ t \in J$.

One can have similar remarks for the inequalities (4.2) and (4.3).

Remark 4.7. Let $0 < \alpha < 1$, if z is solution of the inequality (4.1) then z is a solution of the following integral inequality

$$\left| z(t) - \frac{z_a}{\Gamma(\gamma)} (\psi(t) - \psi(a))^{\gamma-1} - \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} (f(s, z(s), Hz(s)) ds \right| \leq \epsilon \frac{(\psi(t) - \psi(a))^{\alpha}}{\Gamma(\alpha+1)}.$$

Indeed, by Remark 4.6 we have that

$$D_{a_+}^{\alpha,\beta;\psi} z(t) = f(t, z(t), Hz(t)) + g(t), \ t \in J.$$

Then

$$z(t) = \frac{z_a}{\Gamma(\gamma)} (\psi(t) - \psi(a))^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} (f(s, z(s), Hz(s)) + g(s)) \, ds.$$

From this it follows that

$$\begin{aligned} &\left| z(t) - \frac{z_a}{\Gamma(\gamma)} (\psi(t) - \psi(a))^{\gamma-1} - \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} (f(s, z(s), Hz(s)) ds \right| \\ &= \left| \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} g(s) ds \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} |g(s)| \, ds \\ &\leq \epsilon \frac{(\psi(t) - \psi(a))^{\alpha}}{\Gamma(\alpha+1)}. \end{aligned}$$

We have similar remarks for the inequality (4.2) and (4.3).
Now, we give the main results, generalised Ulam-Hyers-Rassias stable results, in this section.

[H3]: There exists an increasing finctions $\varphi \in C_{1-\gamma,\rho}[a, b]$ and there exists $\lambda_\varphi > 0$ such that for any $t \in J$

$$I_{a_+}^{\alpha;\psi} \varphi(t) \leq \lambda_\varphi \varphi(t).$$

Theorem 4.8. *The hypothesis [H1] and [H3] holds. Then Eq.(1.1) is generalised Ulam-Hyers-Rassias stable.*

Proof. Let z be solution of 4.3 and by Theorem 3.2 there x is unique solution of the problem

$$\begin{aligned} D_{a_+}^{\alpha,\beta;\psi} x(t) &= f(t, x(t), Hz(t)), \quad t \in J, \\ I_{a_+}^{1-\gamma;\psi} x(a) &= I_{a_+}^{1-\gamma;\psi} z(a). \end{aligned}$$

Then we have

$$x(t) = \frac{z_a}{\Gamma(\alpha)} (\psi(t) - \psi(a))^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} f(s, x(s), Hx(s)) ds.$$

By differentiating inequality (4.3), we have

$$\left| z(t) - \frac{z_a}{\Gamma(\alpha)} (\psi(t) - \psi(a))^{\gamma-1} - \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} f(s, z(s), Hz(s)) ds \right| \leq \lambda_\varphi \varphi(t).$$

Hence it follows

$$
\begin{aligned}
|z(t)-x(t)| &\leq \left| z(t) - \frac{z_a}{\Gamma(\alpha)} (\psi(t)-\psi(a))^{\gamma-1} - \frac{1}{\Gamma(\alpha)} \int_a^t \psi^{'}(s) (\psi(t)-\psi(s))^{\alpha-1} f(s,x(s),Hx(s))ds \right| \\
&\leq \left| z(t) - \frac{z_a}{\Gamma(\alpha)} (\psi(t)-\psi(a))^{\gamma-1} - \frac{1}{\Gamma(\alpha)} \int_a^t \psi^{'}(s) (\psi(t)-\psi(s))^{\alpha-1} f(s,z(s),Hz(s))ds \right| \\
&+ \left| \frac{1}{\Gamma(\alpha)} \int_a^t \psi^{'}(s) (\psi(t)-\psi(s))^{\alpha-1} f(s,z(s),Hz(s))ds \right. \\
&\left. - \frac{1}{\Gamma(\alpha)} \int_a^t \psi^{'}(s) (\psi(t)-\psi(s))^{\alpha-1} f(s,x(s),Hx(s))ds \right| \\
&\leq \lambda_\varphi \varphi(t) + \frac{L(1+H)}{\Gamma(\alpha)} \int_a^t (\psi(t)-\psi(s))^{\alpha-1} |z(s)-x(s)|\, ds.
\end{aligned}
$$

By Lemma 2.5, there exists a constant $M^* > 0$ independent of $\lambda_\varphi \varphi(t)$ such that

$$|z(t)-x(t)| \leq M^* \lambda_\varphi \varphi(t) := C_{f,\varphi} \varphi(t).$$

Thus, Eq.(1.1) is generalized Ulam-Hyers-Rassias stable. □

Remark 4.9. (i) Under the assumption of Theorem 4.8, we consider (1.1) and the inequality (4.2). One can repeat the same process to verify that Eq.(1.1) is Ulam-Hyers-Rassias stable.

(ii) Under the assumption of Theorem 4.8, we consider (1.1) and the inequality (4.1). One can repeat the same process to verify that Eq.(1.1) is Ulam-Hyers stable.

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